# Mathematical Foundations of Infinitie-Dimensional Statistical Model

Chap.3.7.3 - 3.7.4

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- Definition 3.7.22 Let $X(t), t \in T$, be a bounded process whose finite-dimensional laws correspond to the finite-dimensional projections of a tight Borel probability measure on $l_\infty(T)$, and denote by $\tilde{X}$ a measurable version of $X$ with separable range. Let$X_n(t), t \in T$, be bounded processes. Then we say that $X_n$ converge in law to X in $l_\infty(T)$, or uniformly in $t \in T$, or that

$$X_n \to_L X, \text{ in } l_\infty(T)$$

if

$$E^* H(X_n) \to EH(\tilde{X})$$

for all functions $H : l_\infty(T) \mapsto \mathbb{R}$ bounded and continuous, where $E^*$ denotes outer expectation.

- Theorem 3.7.23 Let $X_n(t), t \in T, n \in \mathbb{N}$ be a sequence of bounded process. Then the following statements are equivalent:

  (a) The finite-dimensional distributions of the processes $X_n$ converges in law, and there exists a pseudo-metric $d$ on $T$ such that $(T, d)$ is totally bounded, and

  $$\lim_{\delta \to 0} \limsup_{n \to \infty} Pr^* \{ \sup_{d(s,t) \leq \delta} |X_n(t) - X_n(s)| > \epsilon \} = 0$$

  for all $\epsilon > 0$

  (b) There exists a process $X$ whose law is a tight Borel probability measure on $l_\infty(T)$ and such that

  $$X_n \to_L X \text{ in } l_\infty(T)$$

- Theorem 3.7.25 Let $(\Omega_n, \mathcal{A}_n, \mathcal{Q}_n) \mapsto l_\infty(T), n \in \mathbb{N} \cup \{\infty\}$, be probability spaces, an let $X_n : \Omega_n \mapsto l_\infty(T)$ be bounded processes such that $X_\infty$ is Borel measurable and has separable range in $l_\infty(T)$. Then $X_n \to_L X_\infty$ in $l_\infty(T)$ if and only if there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathcal{Q}})$ and perfect maps $\phi_n : \tilde{\Omega} \mapsto \Omega_n$ such that $\mathcal{Q}_n = \mathcal{Q} \circ \phi_n^{-1}, n \leq \infty$, and $\lim_{n\to\infty} \|\tilde{X}_n - \tilde{X}_\infty\|_T^* = 0 \quad \tilde{Q}$-a.s as $n \to \infty$ where $\tilde{X}_n = X_n \circ \phi_n$

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Let $X_i$ be an S-valued random variable with law $P$. In this section, we assume that $\mathcal{F}$ consists of P-square integrable functions and that

$$\sup_{f \in \mathcal{F}} |f(x) - Pf| < \infty, \ \forall x \in S \tag{1}$$

With this condition, the centred empirical process based on $\{X_i\}$ and indexed by $\mathcal{F}$.

$$f \mapsto (P_n(\omega) - P)(f) = \frac{1}{n} \sum_{i=1}^{n} (f(X_i(\omega)) - Pf)$$

is a bounded map $\mathcal{F} \mapsto \mathbb{R}$; that is, the centred empirical process $P_n - P$ has all its sample paths bounded, and the results from the preceding section apply to it.

$\nu_n(f) := \sqrt{n}(P_n - P)(f),\ f \in \mathcal{F}$ converges in law to the corresponding finite-dimensional distributions $(G_p(f_1), \ldots, (G_p(f_k))$ of centred Gaussian process $\{(G_p(f) : f \in \mathcal{F}\}$ with covariance that of $f(X) - Pf, f \in \mathcal{F}$, that is,

$$\mathcal{L}(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (f_1(X_i) - Pf_1, \ldots, f_k(X_i) - Pf_k)) \to_w \mathcal{L}(G_p(f_1), \ldots, G_p(f_k)), f_i \in \mathcal{F}, k \in \mathbb{N}$$

where $G_P(f), f \in \mathcal{F}$ is a a centred Gaussian process with the same covariance as the process $\{f(X) : f \in \mathcal{F}\}$,

$$E(G_P(f) G_P(g)) = E[(f(X) - Pf)(g(X) - Pg)] = P[(f - Pf)(g - Pg)]$$

We may refer to $G_p$ as the $P$-bridge process indexed by $\mathcal{F}$

- Definition 3.7.26 The class of functions $\mathcal{F}$ *P*-pre-Gaussian if the $P$-bridge process $G_P(f), f \in \mathcal{F}$, admits a version whose sample paths are all bounded and uniformly continuous for its intrinsic $L^2$-distance $d_P^2(f,g) = P(f-g)^2 - (P(f-g))^2,\ f,g \in \mathcal{F}$
- If $\mathcal{F}$ is $P$-pre-Gaussian, then the pseudo-metric space $(\mathcal{F}, d_P)$ is totally bounded, and the law of $G_P$ is a tight Borel probability measure on the Banach space $C_u(\mathcal{F}, d_P)$

  (By Proposition 2.1.5 and Sudakov's theorem(Cor. 2.4.13)

- Definition 3.7.29 The class of functions $\mathcal{F} \subset L^2(S, \mathcal{S}, P)$ satisfying the boundedness condition (1) is a $P$-Donsker class or that $\mathcal{F}$ satisfies the central limit theorem for $P, \mathcal{F} \in CLT(P)$ for short, if $\mathcal{F}$ is $P$-pre-Gaussian and the $P$-empirical processes indexed by $\mathcal{F}, \nu_n(f) = \sqrt{n}(P_n - P)(f), f \in \mathcal{F}$, converge in law in $l_\infty(\mathcal{F})$ to the Gaussian process $G_P$ as $n \to \infty$

- Theorem 3.7.31 Assume that $\mathcal{F} \subset L^2(S, \mathcal{S}, P)$ and satisfies condition (1). Then the following conditions are equivalent:

(a) $\mathcal{F}$ is a $P$-Donsker class.

(b) The pseudo-metric space $(\mathcal{F}, d_P)$ is totally bounded, and

$$\lim_{\delta \to 0} \limsup_{n \to \infty} Pr^*\Big( \sup_{f,g \in \mathcal{F}, d_P(f,g) \le \delta)} |\sqrt{n}(P_n - P)(f-g)| > \epsilon \Big) = 0$$

for all $\epsilon > 0$

(c) There exists a pseudo-distance e on $\mathcal{F}$ such that $(\mathcal{F}, e)$is totally bounded, and

$$\lim_{\delta \to 0} \limsup_{n \to \infty} Pr^*\Big( \sup_{f,g \in \mathcal{F}, e(f,g) \le \delta)} |\sqrt{n}(P_n - P)(f-g)| > \epsilon \Big) = 0$$

for all $\epsilon > 0$

- Proposition 3.7.32 Let $\mathcal{F} \subset L^2(P)$ be a P-Donsker class satisfying the conditions (2) and let F be its measurable cover. Then.

$$\lim_{t\to\infty} t^2 Pr\{F > t\} = 0$$

. If the P-Donsker class $\mathcal{F}$ only satisfies condition (1), and $\bar{F}$ is the measurable cover of the centred class $\{f - Pf : f \in \mathcal{F}\}$, then

$$\lim_{t\to\infty} t^2 Pr\{\bar{F} > t\} = 0$$

- Proposition 3.7.33 If $\mathcal{F}_1$ and $\mathcal{F}_2$ are P-Donsker classes of functions, then so is $\mathcal{F}_1 \cup \mathcal{F}_2$
- Proposition 3.7.34 If $\mathcal{F}$ is P-Donsker, So is $H(\mathcal{F}, P)$ where $H(\mathcal{F}, P) = \{g : S \mapsto \mathbb{R} : g(x) = \lim_n g_n(x) \ \forall x \in S \quad \text{and} \quad \lim_n P(g_n - g)^2 = 0, g_n \in sco(\mathcal{F})\}$